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RIGHTS IN A THEORY OF GROUP CHOICE: Some Notes

by

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Working Paper No. 69: Rights, Consensus and the Optimum Savings Problem.

Working Paper No. 70: On Trading Personal Liberty.

Working Paper No. 71: Liberty, Fairness and the Possibility of Social Choice.

Working Paper No. 72: Rights and Paretianism in a Framework of Fuzzy Preferences.

IGHTS, CONSENSUS AND THE OPTIMUM SAVINGS PROBLEM

S.Subramanian

ABSTRACT : In the context of an example borrowed from the 'optimum savings' literature, this paper seeks to evaluate the appeal of the restrictions on libertarianism that have been suggested by Gibbard (1974) and Gaertner and Krüger (1981) as responses to, respectively, the 'Gibbard paradox' and the 'Sen paradox' in social choice theory. More generally, the paper points to the importance of -- and complexities induced by -- taking account of private motives behind individual preferences in preference aggregation exercises.

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1. BACKGROUND.

Amartya Sen (1970) has proposed a 'mild' libertarian requirement which it may be deemed desirable for social choices to respect. His 'minimal libertarian' condition assumes the following form: if any two social states differ from each other solely in respect of some feature which may be judged to be a given person's exclusive concern, then that person's preference over the relevant pair of alternatives should be socially respected; and a minimally libertarian restriction on social choices would be that at least two individuals in society should be accorded such a privilege. Sen (*op. cit.*) has established the -- by now, well known -- result of an incompatibility between libertarianism and the (weak) Pareto principle, which seeks a reflection of unanimous preferences

over any pair of social states in the social preference over that pair.

This result can be highlighted with the help of an example due to Sen. Three social states are defined: NR, NN and RN, where the first element of each state describes individual 1's (a prude's) activity, and the second element individual 2's (a lewd's) activity, with R standing for 'reads *Lady Chatterly's Lover*' and N standing for 'does not read *LCL*'. One must now have: NR socially preferred to NN by liberty to the lewd; NN socially preferred to RN by liberty to the prude; and RN socially preferred to NR by the Pareto principle (we are told that the prude and the lewd would, from protective and sadistic concern respectively, both prefer the prude reading the book to the lewd reading it). There is no socially 'best' alternative in the triple of social states {NR,NN,RN} -- and consistent social choice is rendered impossible.

In a recent paper, Gaertner and Krüger -- hereafter G-K -- (1981) have argued that the trouble stems from according libertarian rights to individuals who do not betray what the authors call 'self-supporting' preferences. In the *Lady Chatterly's Lover* example discussed above, neither the prude nor the lewd displays self-supporting preferences: within his protected sphere (NN,RN), the prude manifests a preference for not reading to reading, but when it comes to a choice

between NR and RN, he allows the lewd's activity in the two states to influence him to reverse his own preference for not reading to reading. Similar ponsiderations apply to the lewd's preferences. G-K suggest that in such situations where individuals are not willing to support the preferences they manifest in their respective 'protected spheres' in preferences over other pairs of alternatives, the individuals should forfeit their claim to having their rights respected in social choices. As they put it (K-G (1981, pp.20-21)): '... no individual can have it both ways, namely unconditional social protection of his protected sphere and maximal gains from collective decisions. Social protection has a price which the individual is required to pay, and this also seems fair... Each individual is confronted with the problem of either forfeiting this right through meddling or winning social decisiveness at the price of revealing self-supporting preferences'.

A different problem with a suggested resolution which is similar to but independent of that advanced by G-K is due to Alan Gibbard (1974). Gibbard has demonstrated that if each of two individuals is allowed to dictate on at least two pairs of alternatives each, then it is possible that for some admissible individual preference profiles the libertarian condition may not even be internally consistent. Gibbard motivates his discussion with the help

of an example involving a craven conformist (Mrs. Grundy) and a perverse nonconformist (Gibbard himself). Four social states are defined: WY, WW, YY and YW, where the first element in each state stands for Gibbard's wall colour and the second for Mrs.Grundy's wall colour, with W signifying 'white' and Y signifying 'yellow'. If Mrs.Grundy asserts her libertarian right to match Gibbard's wall colour, and Gibbard his libertarian right to differentiate from Mrs.Grundy, then one would end up with: WW socially preferred to WY by liberty to Mrs.Grundy; WY socially preferred to YY by liberty to Gibbard; YY socially preferred to YW by liberty to Mrs.Grundy; and YW socially preferred to WW by liberty to Gibbard. The social preference sequence is cyclic, and again no consistent social choice is possible.

To get out of this worrisome problem of an internal inconsistency in the libertarian principle, Gibbard (*op. cit.*) proposes that individual preferences in protected spheres need be socially respected only when such individual preferences are *privately unconditional*. In the wall colour example just discussed, neither Gibbard's nor Mrs.Grundy's preferences are privately unconditional: they are contingent on, and vary with, the actions of the other individual. Gibbard suggests that only 'intrinsic' preferences in protected spheres need be respected by society: '... a reasonable libertarian claim would disregard such conditional preferences' (Gibbard(1974, p.393)). In the context of the wall colour example, Gibbard's

proposed restriction is (arguably) reasonable -- but it is not clear that the restriction has universal appeal.[1]

The Gibbardian restriction on individual rights to only those cases in which individuals display privately unconditional preferences in their protected spheres, and the G-K restriction of individual rights to only those cases in which individuals display self-supporting preferences can be fruitfully examined in the context of the 'optimum savings' problem -- a problem made familiar by the writings of (among others) Baumol (1968), Sen (1961, 1967, 1982), Marglin (1963), Lind (1964) and Tullock (1964). Using the savings problem as a sort of spot reference for associated problems subsumed (broadly) under the socalled 'Isolation Paradox', (see Sen (1961,1967)), one can see that the Gibbardian and the G-K restrictions on personal rights can have consequences which do not always accord with the dictates of one's moral intuition. Related issues have been explored by Ben Fine (1974) and Amartya Sen (1974) -- although not in the explicit context of Gibbard's and G-K's restrictions on libertarianism.

We wind up this introductory section with a brief paragraph on concepts and notation. X is a set of all available social states. If P(X) is the power set of X and $\Sigma := P(X)-\{\emptyset\}$, then every $A \in \Sigma$ is an agenda. $H = \{1,\ldots,i,\ldots,/H/\}$ is the set of all individuals in society. Each individual i is assumed to have a preference ordering -- an ordering being

a reflexive, complete and transitive binary relation R_i on X. To each R_i on X there corresponds an asymmetric component P_i. A functional collective choice rule (FCCR) is a functional relation f which, for every agenda A and /H/-tuple of individual preference orderings $\{R_i\}$ on A, specifies a choice function C(.) yielding a nonempty subset C(A) of A:

$C(A) = f(\{R_i\}_{i \in H})$.

C(A) is the set of socially chosen alternatives from A, and is called the choice set of A. The FCCR is decisive whenever the choice set is a singleton ($\neq \{\emptyset\}$). We shall take it that the existence of a decisive FCCR is a necessary condition for the social choice mechanism to be deemed 'successful'. Further strenuous formalism will be avoided since the point sought to be made in this note is a relatively simple one and can be adequately conveyed with the help of allusive examples[2].

2. AN EXAMPLE FROM THE OPTIMUM SAVINGS LITERATURE.

This section draws heavily om Marglin (1963) and Sen (1967). The Marglin-Sen framework can be outlined as follows. Consider a society constituted by two persons who are designated 1 and 2 respectively. Each person is considering the marginal decision of undertaking to save an incremental unit of income. Person i (i =1,2) places a value of unity on a unit of consumption by himself (this is a simple normalization). Person 1 values a unit of consumption

by his contemporary, person 2, at β_{12} (<1), a unit of consumption by his own heirs at α_1 ($> \beta_{12}$), and a unit of consumption by 2's heirs at α_{12} ($< \alpha_1$). The parameters β_{21}, α_2, and α_{21} can be analogously defined for person 2. A proportion $\lambda \in [0,1]$ of each person's saving will go to his own heirs, and a proportion $(1-\lambda)$ to his contemporary's heirs. A unit of saving undertaken (i.e., consumption forgone) today will yield k (> 1) units of consumption in the future.

If we write i's utility function -- assumed increasing and continuously differentiable in each of its arguments -- as $U_i = U_i(C_i, C_j, C_i^f, C_j^f)$, where C_i stands for i's current consumption, C_j for j's current consumption, C_i^f for the consumption of i's heirs in the future, and C_j^f for the consumption of j's heirs in the future, then, taking total differentials, we have:

$$dU_i = \frac{\delta U_i}{\delta C_i}dC_i + \frac{\delta U_i}{\delta C_j}dC_j + \frac{\delta U_i}{\delta C_i^f}dC_i^f + \frac{\delta U_i}{\delta C_j^f}dC_j^f .$$

Noting that $dC_i^f := -\{k\lambda dC_i + k(1-\lambda)dC_j\}$ and $dC_j^f := -\{k\lambda dC_j + k(1-\lambda)dC_i\}$, dU_i can be written as

$$dU_i = \{\frac{\delta U_i}{\delta C_i} - k\{\lambda\frac{\delta U_i}{\delta C_i^f} + (1-\lambda)\frac{\delta U_i}{\delta C_j^f}\}\}dC_i + \{\frac{\delta U_i}{\delta C_j} - k\{(1-\lambda)\frac{\delta U_i}{\delta C_i^f} + \lambda\frac{\delta U_i}{\delta C_j^f}\}\}dC_j$$

If U_i is linear in its arguments over the relevant range,(1) then the partials on the Right Hand Side of the equation above are constants; putting $\frac{\delta U_i}{\delta C_i} = 1$, $\frac{\delta U_i}{\delta C_j} = \beta_{ij}$, $\frac{\delta U_i}{\delta C_i^f} = \alpha_i$ and

$\frac{\delta U_i}{\delta C_j^F} = \alpha_{ij}$, we see that the valuations discussed earlier can be interpreted as the relevant marginal utilities, and (1) can be rewritten as

$$dU_i = \{1-k\{\alpha_i\lambda+\alpha_{ij}(1-\lambda)\}\}dC_i + \{\beta_{ij}-k\{\alpha_i(1-\lambda)+\alpha_{ij}\lambda\}\}dC_j \quad \ldots(2)$$

In what follows, we shall make a small departure from the Marglin-Sen framework by placing the following additional Restriction: $\alpha_1 = 0$ for $dC_1 = 0$.

This means that person 1 places no value at all on his own heirs' consumption when he himself contributes nothing to the consumption of the future generations: he feels, as it wre, entitled to take only when he gives. (Note, however, that (i) person 1 values the consumption of 2's heirs the same way, whether or not 2 contributes to the consumption of future generations; and (ii) the restriction stated above applies -- without loss of generality -- only to person 1).

Now each person can either decide to save an additional unit (S), or not save (N). Four possible social states can be defined: NN(neither person saves), SN(only person 1 saves), SS(both persons save), and NS(only person 2 saves). Equation 2) and the restriction stated above are helpful in deriving each person's net utility gain dU_i (i=1,2) under each of the four possible social states, and this information is presented in the pay-off matrix below:

Pay-off Matrix

dC_1 \ dC_2	0		-1	
	Social state; NN		Social state: NS	
	dU_1	dU_2	dU_1	dU_2
0	0	0	$k\lambda\alpha_{12} - \beta_{12}$	$\{(1-\lambda)\alpha_{21}+\lambda\alpha_2\} xk - 1$
	Social state: SN		Social state: SS	
	dU_1	dU_2	dU_1	dU_2
-1	$\{\lambda\alpha_1+(1-\lambda) x\alpha_{12}\}k - 1$	$\{\lambda\alpha_{21}+(1-\lambda) x\alpha_2\}k - \beta_{21}$	$(\alpha_1+\alpha_{12})k-(1+\beta_{12})$	$(\alpha_2+\alpha_{21})k - (1+\beta_{21})$

A possible set of numerical values for the parameters λ, β_{12}, β_{21}, α_1, α_2, α_{12}, α_{21} and k could be:
$\lambda=3/4$, $\beta_{12}=1/3$, $\beta_{21}=1/100$, $\alpha_1=1/2$ (except when only person 2 saves, in which case $\alpha_1=0$), $\alpha_2=1/2$, $\alpha_{12}=1/4$, $\alpha_{21}=1/160$, and $k=2$.

Plugging the appropriate numerical values of the parameters into the expressions for dU_i $(i=1,2)$ in the pay-off matrix above, the net utility gain to each person under the four social states can now be written as:

Social state	dU_1	dU_2
NN	0	0
NS	1/24	-79/320
SN	-1/8	399/1600
SS	1/6	1/400

In terms of the net utility gain to each person under each state, the preference rankings of the four states by the two individuals would be as follows:

Person 1	Person 2	
SS	SN	Descending order of preference ↓
NS	SS	
NN	NN	
SN	NS	

It should be mentioned here that the preference configuration presented above does not correspond to any of the three configurations discussed, in a useful taxonomy, by Sen (1974), and which he calls, respectively, PD (Prisoner's Dilemma), AG (Assurance Game) and OR (Other Regarding) preference configurations. The PD, AG and OR preference configurations, in the context of the optimum savings problem, would correspond, respectively, to the profiles {(NS,SS,NN,SN),(SN,SS,NN,NS)}, {(SS,NS,NN,SN),(SS,SN,NN,NS)} and {(SS,SN,NS,NN),(SS,NS,SN,NN)}. In the preference configuration used in this note -- {(SS,NS,NN,SN),(SN,SS,NN,NS)} -- one of the persons (person 1) betrays AG preferences, while the other (person 2) reveals PD preferences. It is this 'mixed' profile which -- to anticipate -- leads to unsatisfactory social outcomes when the Pareto principle is combined with Gibbard libertarianism or G-K libertarianism. To this problem, we now turn.

3. THE LIBERTY PARADOX REVISITED.

Given the configuration of individual preferences

over the designated agenda $A = \{NN, SN, SS, NS\}$, an FCCR which respects libertarianism and the Pareto principle would dictate that:

$NN\ P_1\ SN \rightarrow SN \notin C(A)$ by liberty to 1;

$SS\ P_1\ NS$ and $NN\ P_2\ NS \rightarrow NS \notin C(A)$ by liberty to 1 and 2 respectively;

$SN\ P_2\ SS \rightarrow SS \notin C(A)$ by liberty to 2; and

$SS\ P_1\ NN$ and $SS\ P_2\ NN \rightarrow NN \notin C(A)$ by Pareto.

Thus, $C(A) = \emptyset$, and the familiar Paretian-libertarian conflict emerges.

In the context of the Lady Chatterly's Lover example, Sen (1976,1979) has argued persuasively in favour of axing the Pareto principle. While the outcome of the prude reading the book and the lewd not reading it is a social optimum, one can argue that it is a poor kind of social optimum which is achieved by each person's desire to cut his own nose in order to spite the other's face. A similar kind of motivational problem however does not characterize the Pareto ranking of SS over NN in the context of the savings problem. Indeed -- and as is well known -- in the context of situations such as those of the Prisoner's Dilemma and the Assurance Game, the Pareto criterion favours the emergence of a social contract the outcome of which, in some straightforward sense, is superior to the outcome corresponding to an insistence on atomistic rationality.

(On this, see in particular Sen (1974)). If we decide to preserve the Pareto principle, are there grounds for violating the liberty principle ?

This is where Gibbard's and G-K's proposed restrictions on the liberty principle would come into play. Note, first, that person 1 -- unlike person 2 -- is <u>not</u> displaying privately unconditional preferences in his protected spheres. When 1's assigned pair is (NN,SN), he reveals a preference for not saving to saving, but reverses this preference when his assigned pair is (SS,NS). By contrast, whether person 2's assigned pair is (NN,NS) or (SN,SS), he displays a uniformly consistent preference for not saving to saving. By Gibbard's proposed restriction of individual rights to only those cases in which persons betray privately unconditional preferences in their respective protected spheres, person 1 should forfeit his libertarian claims. We would then have:

$SN\ P_2\ SS \rightarrow SS \notin C(A)$ by Gibbardian liberty to 2;

$NN\ P_2\ NS \rightarrow NS \notin C(A)$ by Gibbardian liberty to 2; and

$SS\ P_1\ NN$ and $SS\ P_2\ NN \rightarrow NN \notin C(A)$ by Pareto.

This leaves us with: $C(A) = \{SN\}$.

Is this a satisfactory outcome ? Arguably not, and for at least two reasons:

(a) From the numerical values assigned to the parameters in the savings problem, it is immediately discernible that person 1 has a greater concern for the consumption of person 2 and his heirs than 2 has for the consumption of 1 and 1's heirs.

(Note that α_{12} (=1/4) > α_{21} (=1/160) and β_{12} (=1/3) > β_{21} (=1/100)). In some perfectly straightforward sense, then, person 1 is less 'selfish' or more 'other regarding' (in a benign way) than person 2 is; yet, it is person 1's liberty which is being sought to be constrained.

(b) A related point is that -- again in some direct sense -- 1's libertarian claim seems to be more flexible and 'reasonable' than 2's. Person 1's libertarian claim is the relatively weak one: 'I assert my right to abstain from saving when every one else abstains, but am willing to waive this right if others are willing to save, too', while 2's claim is the relatively hard-nosed one: 'Irrespective of whether others save or not, I assert my right to abstain from saving'. Under the Gibbardian restriction, 2's consistency is rewarded and 1's inconsistency is punished -- without regard for the possibility that the latter's inconsistency is more public-spirited than the former's consistency.

What, next, of the G-K restriction on libertarianism? Notice that neither person 1 nor person 2 betrays self-supporting preferences. While 2 shows a consistent preference for not saving to saving in his assigned pairs (SS,SN) and (NN,NS), when it comes to a choice between SS and NN, he betrays a preference for not saving to saving. As for person 1, when his assigned pair is (SS,NS), he prefers saving to not saving, but reverses this preference over the pair (NS,SN); similarly, when his assigned pair is (NN,SN),

he prefers not saving to saving, but reverses this preference over the pair (SS,NN). Neither person, by the G-K restriction, can claim unconditional protection of his libertarian demands. We would then have:

SS P_1 NN and SS P_2 NN → NN $\notin$ C(A) by Pareto; and

SS P_1 NS and SS P_2 NS → NS $\notin$ C(A) by Pareto, leaving us with:

C(A) = {SS} or {SN}.

A feature in favour of the G-K restriction on libertarianism is that it does not rule the social optimum SS out of the choice set. Unfortunately, the alternative SN is not ruled out either, and the difficulties with seeing a state in which only person 1 saves as the appropriate social choice have alrady been discussed. Insofar as one can choose to regard a bottle as either half-full or half-empty, the G-K restriction is not wholly satisfactory.

The results of this section are briefly discussed in the next one.

4. CONCLUDING DISCUSSION.

In this note, we have briefly reviewed two well-known paradoxes in collective choice theory: the Sen paradox(of an impossible Paretian libertarian) and the Gibbard paradox (of an impossible libertarian libertarian). In order to secure a possibility result, it is sufficient to deny (or suitably modify) either the Paretian claim or the libertarian claim. A question which arises in this

context is whether or not the motivations underlying individual preferences are relevant to a decision on which claim is admissible and which not. Sen (1974, 1976, 1979) has consistently argued in favour of evaluating motivations underlying individual preferences; so too Gibbard, but with less consistency[3]. Arguing from a specific example Gibbard has suggested a way out of his paradox, which consists in denying libertarian rights in situations in which people do not betray privately unconditional preferences. Gaertner and Krüger, while not disagreeing with the notion of incorporating motives into the formal theory feel however that: 'As long as the formal framework (i.e. the set of individual orderings) provides too little information on the motives behind each person's preferences, this evaluation remains a gratuitous piece of informal interpretation' [G-K (1981, p.19)]. Like Gibbard, G-K suggest a restriction on the libertarian claim which amounts to denying rights to individuals who do not reveal self-supporting preferences; unlike Gibbard, G-K do not rationalize this restriction by an appeal to individual motivation.

It is interesting to note that in the context of the Lady Chatterly's Lover example, it is precisely the invocation of motivations behind individual preferences which presents a case in favour of axing the Pareto principle (Sen's suggested resolution) and the libertarian principle (G-K's suggested resolution)[4]. Likewise, in the context

of the wall colour example, it is again the issue of private motivations which disposes one favourably to Gibbard's libertarian restriction. In a different framework, however -- such as in the context of the optimum savings problem that we have discussed -- we find that neither Gibbard's libertarian restriction, nor G-K's libertarian restriction, nor yet a restriction of the Pareto principle yields an intuitively acceptable ethical resolution of the Pareto-liberty conflict: the motivations underlying individual preferences are, in this case, uncomfortably non-conducive to any of these approaches to resolution.

Indeed, if we reject both the Gibbard and the G-K restrictions on libertarianism, and reject also a denial of the Pareto principle, we could still argue a case in favour of a possibility result along the following lines. It could be contended that under the circumstances it is least objectionable that person 2 -- as the less public-spirited of the two individuals -- is the one whose rights should be curtailed. We would then have:

$SS \; P_1 \; NS \rightarrow NS \notin C(A)$ by liberty to 1;

$NN \; P_1 \; SN \rightarrow SN \notin C(A)$ by liberty to 1; and

$SS \; P_1 \; NN$ and $SS \; P_2 \; NN \rightarrow NN \notin C(A)$ by Pareto.

We are left with: $C(A) = \{SS\}$.

It is possible to argue that if the realization of the social optimum SS can be purchased at the price of restricting a relatively rigid and selfish person's rights,

the price is not too steep. The point, however, is a debatable one -- and not one which lends itself to dogmatic support. On the other hand, this resolution -- it could be maintained -- is not intrinsically less acceptable than resolutions based on denial of the Pareto principle, or denial of person 1's rights (Gibbard's resolution) or denial of both persons' rights (G-K's resolution).

If the traditional collective choice framework of a 'constitution' is accepted, then motivations underlying individual preferences could be an important informational ingredient in arriving at social decisions and judgements -- although not necessarily in the constructive sense of transforming impossibility results into ethically satisfactory possibility results. Similar individual preference configurations could conceal widely differing underlying individual motivations -- and mechanical rules of aggregation could be less than helpful. Further, rules which are helpful in one context may not be so in other contexts. Finally, even when we do take account of personal motivations -- as in the case of the savings problem discussed in this note -- no clear cut guidelines for arriving at a morally unambiguous resolution of conflicting ethical principles may be available.

The utility of this note must be judged only in terms of the general utility of raising issues without necessarily resolving them.

NOTES.

1. For an interesting discussion of this problem, see Wriglesworth (1985).

2. In particular, we desist here from furnishing tedious formal definitions of libertarianism, paretianism, privately unconditional preferences, and self-supporting preferences: the context in which these concepts are invoked in the text should make their meaning clear.

3. As pointed out by Wriglesworth (1985), Gibbard tends to regard private motivations as crucial in the context of the Gibbard paradox, but questions the appropriateness of taking these into account in the context of the Sen paradox.

4. Despite G-K's explicitly stated position that the '... evaluation [of motives] remains a gratuitous piece of informal interpretation' (G-K (1981,p.19)), there is a strong suggestion that their libertarian restriction stems from generalizing the motives behind individual preferences from a single example (the LCL example): '... each individual is confronted with the *problem* of forfeiting [his] right through meddling ...' (G-K(1981,p.21;emphasis added)). The somewhat *pejorative* characterization of individual interdependences as being invariably motivated by meddlesomeness seems to be at the heart of the G-K *restriction on libertarianism*.

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ON TRADING PERSONAL LIBERTY

S.Subramanian

ABSTRACT: Sugden (1978) has suggested the implementation of 'voting rules' as a resolution of Sen's (1970) paretian-liberal dilemma. But incentive-compatible considerations may rule out the possibility of a non-null core for the 'voting game'.

'And what', said Percy, 'is there in it for me?'
—P.G.Wodehouse: Frozen Assets.

1.INTRODUCTION.

One widely known resolution of Amartya Sen's (1970) result on the 'impossibility of the Paretian liberal' is Alan Gibbard's (1974) scheme of voluntary rights-waiving whereby people are enabled to enter into contracts to 'trade' their rights away in favour of Pareto-undominated outcomes. A second suggested resolution is that of Robert Sugden(1978), who advocates the implementation of 'voting rules' that are designed to guarantee to individuals what he calls 'freedom of choice'; the suggestion takes the form of acknowledging the right of people to, again, enter into negotiated settlements on how each person should vote with a view to ensuring the emergence of a Pareto-undominated alternative as the socially chosen outcome. Both Gibbard's and Sugden's approaches can be seeb as specific recommendations that are consistent with the distinguished libertarian tradition of free contract.

In a recent paper, Kaushik Basu (1984) has pointed to a problem of incentive-compatibility in the Gibbardian solution, and this problem is relevant also for Sugden's solution. In a broad way, these incentive-compatible considerations would seem to cast serious doubts on the espousal of trade as a vehicle for freedom of choice. In this paper, I discuss Sugden's resolution of the Pareto-liberty problem in the light of Basu's critique of the Gibbardian solution.

This paper is organized as follows. Section 2 contains a very abbreviated version of Sugden's critique of Sen's formulation of the liberty principle. In section 3 certain formal definitions of concepts relevant to Sugden's suggested framework for dealing with the notion of individual liberty in social choice are presented. Section 4 is concerned with an examination of Sugden's definition of the 'core of a voting game': it is argued that this definition fails to take account of a problem of incentive-compatibility, and when allowance is made for this, one encounters again a 'liberty paradox' -- but this time within the context of Sugden's own modified framework. In section 5, an informal discussion of the main point made in section 4 is offered. Section 6 contains concluding observations.

2. SUGDEN ON LIBERALISM: A SUMMARY.

Sugdem, in the work cited earlier, has provided an extensive

critique of Sen's 'impossibility of a Paretian liberal' result, in which he questions the fruitfulness of seeking to formalize meaningfully the notion pf individual liberty within a conventional social choice-theoretic framework that relies on the Arrovian conception of a 'constitution'. An Arrovian constitution, Sugden argues, is '...simultaneously a decision-making procedure and the source of a value system'[1] in the hands of a public official who is '... a dictator -- however benevolent his intentions'[2]. Within this 'paradigm of a dictatorial decision-maker'[2], the 'existence of coercion is axiomatic'[3]. Consequently, it is not meaningful to speak of liberty, as Sen does, in terms of a requirement that for each individual there should exist a pair of social states such that the individual's preference over that pair is socially respected; the implication of such a requirement would be that if an individual preferred some state x to some state y, then he '... should be forced not to [choose y]'[4] -- an expression which, in the context of the Arrovian framework, can be used interchangeably with the more appealing expression 'he should not be forced to choose y'. Briefly, the Arrovian paradigm of collective choice fails to allow for the fundamentally libertarian requirement of <u>freedom of choice</u>. Sugden argues for a framework which will provide for the implementation of freedom of choice and this, in his view, would call for the decision-maker's putting into

effect some voting rule.

3. VOTING RULES, FREEDOM OF CHOICE AND THE UNANIMITY PRINCIPLE.

A quick review of certain formal definitions pertinent to Sugden's framework is in order.

First, a decision-maker's action is defined to be any ordred pair (Q, f^Q) where Q is a non-empty delegated set of social states, and f^Q is a strict voting rule for the set Q. If $H = \langle 1, \ldots, i, \ldots, /H/ \rangle$ is the set of all individuals im society, Σ^Q is the set of all non-empty subsets of Q, and V^Q is the set of all possible voting profiles on Q, then by a voting rule for the non-empty set Q is meant a functional relation $f^Q: V^Q \rightarrow \Sigma^Q$, such that for every /H/-tuple of individual votes $\langle v_i^Q \rangle$, f^Q specifies a non-empty subset C^Q (of Q) which is the set of socially chosen elements from Q. f^Q will be said to be a strict voting rule, if (i) C^Q is a singleton ($\neq \{\emptyset\}$); and (ii) with each state in the set Q there must be associated at least one set of individuals' votes.

A decision-maker's action (Q, f^Q) will be said to give a coalition K (with $/K/ \geq 1$) of individuals freedom of choice with respect to the set T if there exists a /K/-tuple of individual votes $\langle v_i^Q \rangle$ such that, given this configuration of individual votes, and no matter how those not in K vote, it is the case that $C^Q \subseteq T$. Sugden elucidates with the following example: '... suppose that [T] is the

set of all states which have the property that individual i is married to individual j. Then to say that the set <i,j> has freedom of choice with respect to the set [T] is to say that i and j are free to marry if they both choose to do so'.[5]

A liberal, for Sugden, is one who would require of the decision-maker's action (Q, f^Q) that it be such as to give '... certain specially-valued freedoms of choice to particular individuals or groups'.[6]

By the unanimity principle -- which is the analogue, in Sugden's framework, of the Pareto principle in the conventional social choice framework -- is meant that the set of all individuals in society should have freedom of choice with respect to every social state.

Sugden contends that there is no contradiction between unanimity and liberalism as defined above; in particular, if all individuals are invited to vote openly in the light of their respective preference orderings over a given set of social states, and in such a way as will permit them to engage in mutual bargaining about how each should vote, then the outcome (i.e., the socially chosen alternative) of this 'voting game' will be in the 'core' of the game. By the core of the game is meant the following. First, a state $q \varepsilon Q$ will be said to be viable with respect to the coalition of individuals K if and only if there does not

exist a set T of social states such that (i) $\forall x \varepsilon T$, $\forall i \varepsilon K$: $xP_i^Q q$; and (ii) the coalition K has freedom of choice with respect to the set T. (Note that P_i^Q is the asymmetric part of individual i's 'weak' preference ordering R_i^Q on Q). A state $q \varepsilon Q$ will be said to be in the core S(Q) of the game if and only if q is viable with respect to all coalitions.

4. A PROBLEM OF INCENTIVE COMPATIBILITY.

Sugden's definition of the core of a voting game takes account only of a necessary condition -- that of q having to be viable with respect to all coalitions -- which however is not sufficient. In particular, it seems to be reasonable to impose the additional requirement that there be no incentive for any coalition of individuals to enter into a mutually beneficial trade of their votes which results in a configuration of individual votes the outcome of which is preferred by the coalition to the outcome q. To build this requirement more formally into the definition of the core of a voting game, consider the following.

Let L be some coalition of individuals, with $1 \leqq /L/ \leqq /H/$. Let V^{*Q} be the set of all voting profiles $\langle v_i^{*Q} \rangle_{i \varepsilon H}$ for which q is the socially chosen alternative. Then:

Definition: A state $q \varepsilon Q$ will be said to be in the core S(Q) of the voting game if and only if

(i) q is viable with respect to all coalitions; and

(ii) there do not exist a coalition L of individuals, a /H/-tuple of individual votes $\langle v_i^{*Q}\rangle \varepsilon V^{*Q}$ and a /L/-tuple of individual votes $\langle v_i^{!Q}\rangle$ ($\neq \langle v_i^{*Q}\rangle_{i\varepsilon L}$) such that, for all $j\varepsilon L$,

$$f^Q(\langle v_i^{*Q}\rangle_{i\varepsilon H\setminus L} \cup \langle v_i^{!Q}\rangle_{i\varepsilon L}) P_j^Q f^Q(\langle v_i^{*Q}\rangle_{i\varepsilon H}).$$

Sugden's suggested resolution of the liberty paradox is based on the presumption that whenever condition (i) in the definition above is satisfied, so is condition (ii). The presumption is, in general, wrong, as will be shown with the help of an example adapted from Gibbard (1974).

Three social states are defined: j is a state in which Angelina weds the judge; e is a state in which Angelina weds Edwin; and s is a state in which Angelina, Edwin and the judge all remain single. The story has it that Angelina would like most to marry Edwin, failing which she would prefer marrying the judge to remaining single. Edwin would like best not to be encumbered with marriage, but between having Angelina marry the judge and marry himself, he would prefer her to marry himself. As for the judge, he would prefer most that Angelina married him, next best that she married no-one, and least of all that she married Edwin.[7] Briefly, Angelina's ranking of the three states is the strict order e, j, s; that of Edwin is the strict order s, e, j; and that of the judge is the strict order j, s, e.

What now are the kinds of freedoms which a liberal

would like to offer Angelina, Edwin and the judge? A reading of some of the examples discussed in Sugden's paper seems to suggest the following scheme of freedoms-allocation. (In what follows, A, E and J will stand for Angelina, Edwin and the judge respectively). The coalition <A> has freedom of choice with respect to the sets <s>, <s,e> and <s,j> -- i.e., Angelina is free to remain unmarried, she is free to resist coercion into marriage with the judge and she is free to resist coercion into marriage with Edwin. The coalition <E> has freedom of choice with respect to the set <s,j> -- i.e., Edwin is free to remain unmarried. The coalition <J> has freedom of choice with respect to the set <s,e> -- i.e., the judge is free to opt for bachelorhood. Finally, the coalitions <A,J> and <A,E> have freedom of choice with respect to the sets <j> and <e> respectively -- i.e., Angelina and the judge are free to choose to marry each other, and Angelina and Edwin, likewise, are free to choose to marry each other.

Let $\langle v_A, v_E, v_J \rangle$ be some combination of votes of individuals A, E and J respectively, with

$v_A = 0$ implying that Angelina wishes to remain single;
$v_A = 1$ implying that Angelina wishes to marry Edwin;
$v_A = 2$ implying that Angelina wishes to marry the judge;
$v_E = 0$ implying that Edwin wishes to remain single;
$v_E = 1$ implying that Edwin wishes to marry Angelina;

v_J = 0 implying that the judge wishes to remain single; and v_J = 1 implying that the judge wishes to marry Angelina. If voting is <1,1,0> or <1,1,1>, the socially chosen state will be e. If voting is <2,0,1> or <2,1,1>, the socially chosen state will be j. In all other cases, the socially chosen alternative will be s.

The question now is: is any of the three possible outcomes -- j, s and e -- of the voting game in its core S(<j,s,e>)? Consider first the outcome j. We can identify a coalition K = <A,E> and a set T = <e> with respect to which K has freedom of choice, such that every member of K prefers each element of T to the chosen state j. By definition, then, j∉S(<j,s,e>). Consider next the outcome s. Again, we can identify a coalition K = <A,J> and a set T = <j> with respect to which K has freedom of choice such that every member of K prefers each element of T to the socially chosen state s. Consequently, s∉S(<j,s,e>). Consider finally the outcome e. It is true that condition (i) of the *definition* of the core of a game is satisfied: we cannot identify any coalition K and set T with respect to which K has freedom of choice such that every member of K prefers each element of T to the alternative e. But what of condition (ii) of the *definition*? Note that there are two possible voting profiles -- <1,1,0> and <1,1,1> -- for which the socially chosen outcome is e. Whichever is the voting profile, ~~we can identify a coalition L = <E> such that if this coalition~~

changes its vote from 1 to 0, the socially chosen alternative (s) corresponding to the revised voting profile will be preferred by the coalition to e. Thus, $e \notin S(\langle j,s,e \rangle)$. Briefly, $S(\langle j,s,e \rangle) = \phi$, and an impossibility result re-emerges.

It may be added that <u>every</u> one of the alternatives j, s, e fails to qualify for being in the core $S(\langle j,s,e \rangle)$ of the voting game by condition (ii) of the <u>definition</u> of the core. This is easily discernible from the tabulated presentation below in which it is shown that for each of the states j, s and e and the voting profile(s) associated with the respective state, there exists a coalition of one or more individuals such that the members of the coalition have an incentive to modify their votes in such a way that the socially chosen alternative corresponding to the revised voting profile is preferred by the members of the coalition to the alternative corresponding to the original (unrevised) voting profile(s). The tabulation is self-explanatory.

Socially chosen outcome	Voting profile associated with socially chosen outcome	A coalition whose members have an incentive to change their votes from those shown in the voting profile in (2)	Revised voting profile which would ensue if members of the coalition identified in (3) were to change their votes	Socially chosen outcome corresponding to revised voting profile in (4), and which is preferred by the members of the coalition in (3) to the alternative in (1)
(1)	(2)	(3)	(4)	(5)
j	<2,0,1>	<A,E>	<1,1,1>	e
j	<2,1,1>	<A>	<1,1,1>	e
s	<0,0,0>	<A,J>	<2,0,1>	j
s	<0,0,1>	<A>	<2,0,1>	j
s	<0,1,0>	<A>	<1,1,0>	e
s	<0,1,1>	<A>	<1,1,1>	e
s	<1,0,0>	<A,J>	<2,0,1>	j
s	<1,0,1>	<A>	<2,0,1>	j
s	<2,0,0>	<J>	<2,0,1>	j
s	<2,1,0>	<J>	<2,1,1>	j
e	<1,1,0>	<E>	<1,0,0>	s
e	<1,1,1>	<E>	<1,0,1>	s

The essence of the problem is brought out in the following very informal account of what 'really' happened when Angelina, Edwin and the judge met to negotiate a contract on how each should vote. The story reveals how the prospect of Holy Wedlock could easily degenerate into the reality of what A.P. Herbert called Unholy Deadlock.

5. A CAUTIONARY TALE.

Edwin opened negotiations by addressing the judge. 'Look here, judge', he said, 'let's come to an agreement. I will promise to vote 0 if you'll promise to vote 0. With both of us voting 0, it won't matter what Angelina votes: we'll all remain single -- and nothing could be more alright with me than that. Of course, if you're not agreeable, I'm prepared to enter into a contract with Angelina whereby both of us will vote 1 -- and _then_ where would you be? Watching Angelina and me head for the altar, which would be a pretty sight worse for you (and for me) than if we were all to remain single. So: will you come with me?'

'One moment, Edwin', said the judge. 'Let me speak to Angelina. Angelina, my dear, you heard what Edwin has just said. I _could_ do as Edwin says, you know, but then it would break my heart to see you remain unmarried, and _you_ know that that's the last thing you'd want to be. So let's see if we can't avoid that. Let me suggest that if you'll promise to vote 2, I'll promise to vote 1. Putting

it differently', the judge concluded gallantly, 'how about it?'

'You heard that, Edwin', murmured Angelina demurely, 'shall I marry the judge -- which I know you can't bear to see -- or will you' (in a voice of steel) 'be reasonable and promise to vote 1 if I vote 1 ?'

'That's a low trick, Angelina', replied Edwin coldly. 'Very well then, if that's the way you want to play the game, that's the way we'll play it --'

'Just a minute, you young fool', roared the judge, 'how about your original offer ? Wait, let's make that deal now. You go ahead and vote 0, and I'll promise to do likew --'

'Judge dearest', broke in Angelina breathlessly, ' would you really do that ? Would you leave me all alone in this harsh, unfriendly world ? Have you no feeling ? Please, please say you'll promise to vote 1 if I promise to vote 2 ?'

'Stop', yelled Edwin, and turned desperately to the judge. 'Quick, let's both agree to vote 0 and end this matter here and now.'

'Oh ?' said the judge suavely, 'And haven't you been listening to Angelina ? She has just conferred on me the greatest honour any woman could bestow on a man. Angelina, I shall be proud to do as you say.'

'Thank you, judge, thank you', whispered Angelina brokenly, adding bravely as she turned to Edwin: 'I thought

we might have spared orselves this, Edwin, but since you insist on remaining adamant --'

'No, no, wait', said Edwin hoarsely, 'anything but that, Angelina. I'll vote 1 if you'll do likewise. Let's end this maddening business.'

'What ? and forfeit forever a life of bachelor independence?' The judge sounded incredulous. 'Come, come, Edwin my boy, you're overwrought, and not thinking sensibly. Let's both opt for bachelorhood, let's both vote 0 --'

'I'll make you a good wife, judge', broke in Angelina hysterically.

'Over my dead body!' shouted Edwin. 'You're not going to marry that tricky old coot, even if it means my having to marry you.'

At this point, the judge called Edwin by an opprobrious name. One hesitates, from this point on, to furnish a detailed account of the rest of the proceedings. Suffice it to say that when the police eventually arrived, they discovered Edwin breathing strenuously from the exertion of having just finished throttling a blue-faced judge, while Angelina, with her hands clapped over her ears, was cleaving the air with ear-splitting screams. Edwin, of corse, was hanged, and Angelina died promptly of shock.

It is regrettable to have to introduce this melancholic note in an otherwise cheerful essay, but the story contains a warning: that resolutions of liberty paradoxes which are

based on the liberal tradition of free contract are liable to be rather more radically conclusive than they were intended to be.

6. CONCLUDING OBSERVATIONS.

The result discussed in this paper is not a very surprising one in the light of a problem of incentive-compatibility pointed out by Basu (1984) in the context of Gibbard's (1974) scheme of 'rights-waiving' as a resolution of the original Sen paradox: both Sugden's and Gibbard's resolutions draw on the liberal tradition of free contract. The difficulty however is that even in a situation in which agents are operating with complete knowledge of one another's preferences, the possibility of emergence of a sustainable contract may break down. The problem with the Sen dilemma was the non-existence of a 'best' alternative; the problem with the Sugden/Gibbard approach is the non-existence of a 'best' voting strategy/rights-waiving strategy for the players participating in the game.

It would appear that Sen's impossibility result is rather like the bad penny. It has a tendency to keep turning up.

NOTES.

1. Sugden (1978), p.248-49.
2. Ibid., p.250.
3. Ibid., p.251.
4. Ibid., p.259.
5. Ibid., p.260.
6. Ibid., p.262.

7. The example here deviates from Gibbard's original example in that in the latter, the judge has no freedom of choice: he is happy to do whatever Angelina wants.

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LIBERTY, FAIRNESS AND THE POSSIBILITY OF SOCIAL CHOICE

S.Subramanian

ABSTRACT: This paper demonstrates that Varian's (1974-75) principle of fairness militates, in a specific manner, against Sen's (1970) principle of *liberty*.

1. INTRODUCTION.

In this note, an incompatibility is established between Sen's (1970) principle of liberalism and a principle of allocative *fairness* which has, in the literature, been discussed largely in the context of characterising general equilibrium outcomes. Perhaps the simplest description of 'fairness' is to be found in Varian (1974-75): '(T)o detrmine whether an allocation is equitable, we have only to present each agent with the consumption bundle held by each of the other agents to see whether any agent would wish to exchange his bundle for another agent's bundle. If not, the allocation is equitable....(T)here is no reason for this allocation to be pareto efficient...The properties of equity and efficiency are both desirable...(A)n allocation that has both of these properties... will be called a *fair* allocation'.

2. NOTATION AND CONCEPTS.

X is a set of all possible *social states*, and H = {1,...,i,...,h} a set of all individuals in society. Each $i \in H$ has a *preference ordering* R_i on X whose asymmetric and

symmetric parts are P_i and I_i respectively. A collective choice rule (or CCR) is a functional relation which maps every h-tuple of individual preference orderings $\{R_i\}$ to a unique social preference relation R whose asymmetric and symmetric parts are denoted by P and I respectively. An acyclic social decision function (or ASDF) is a CCR whose range is confined to acyclic social preference relations. One restriction that can be imposed on an ASDF is that of Unrestricted Domain (Condition U): The domain of the ASDF should include every logically possible $\{R_i\}_{i \in H}$.

A second restriction one may wish to impose is that of Minimal Liberalism (Condition L^*) (quoted from Sen(1970)): 'There are at least two individuals such that for each of them there is at least one pair of alternatives over which he is decisive, that is, there is a pair of x,y, such that if he prefers x(respectively y) to y(respectively x), then society should prefer x(respectively y) to y(respectively x)'. A third possible restriction on an ASDF is that of fairness. For a precise statement of this condition, Gibbard's approach to formalising Sen's(1970) characterisation of a social state as 'a complete description of society including every individual's position in it' is useful. To this end we invoke two primitives: 'feature' and 'feature altrnative'. A feature is an object such as 'reading habit', 'sleeping posture', etc. If the feature under consideration is 'sleeping

posture', then a set of feature alternatives corresponding to this feature would include elements such as 'sleeping on back', 'sleeping on belly', etc. Let $B = \langle \beta^1, \ldots, \beta^k, \ldots, \beta^{/B/} \rangle$ be a set of features, with $/B/ \geqq h$. (Note that $/B/$ stands for the cardinality of the set B). Since each feature can be decomposed into a collection of feature alternatives, we can write:

$\beta^k = \langle \beta_1^k, \ldots, \beta_t^k, \ldots, \beta^k_{/\beta^k/} \rangle$, $k=1,2,\ldots,/B/$. If Q is the Cartesian product of the β^k, i.e., $Q = \beta^1 \times \ldots \times \beta^k \times \ldots \times \beta^{/\beta/}$, then the set X of all social states will be given by the h-fold Cartesian product of Q, viz., $X = Q^h$. A social state $x \varepsilon X$ can now be interpreted as a particular configuration (or 'allocation') of feature alternatives of each feature of the world -- one feature alternative for each feature -- as ordained for each individual in society. x can now be written as a list: $x = \langle x_1, \ldots, x_i, \ldots, x_h \rangle$, where for all $i \varepsilon H$, x_i describes individual i's position in x. (Note that individual i's position in x -- x_i -- can itself be written as a list: $x_i = \langle g_i^1, \ldots, g_i^k, \ldots, g_i^{/B/} \rangle$, with $g_i^1 \varepsilon \beta^1, \ldots, g_i^k \varepsilon \beta^k, \ldots, g_i^{/B/} \varepsilon \beta^{/B/}$).

Now, given a social state $x = \langle x_1, \ldots, x_i, \ldots, x_h \rangle$, let $\Sigma(x)$ be a set of all possible permutations of the elements of x. Then a state $x \varepsilon X$ will be said to be <u>equitable</u> if, and only if,

$\sim\{\exists \sigma(x) \varepsilon \Sigma(x) \,\&\, i \varepsilon H : \sigma(x) P_i x\}$.

It is important to note that person i is comparing $\phi(x)$ with x only in terms of his own tastes: while Varian suggests that the notion of fairness draws on Arrow's (1967) conception of 'extended sympathy', Sen (1977) is careful to note that '...no genuine interpersonal comparisons are involved, since each person is assumed to be judging the other person's commodity bondles _in terms of his own tastes_.'

Next, a state $x \in X$ is _pareto optimal_ if, and only if, $\sim\{\exists y \in X: \forall i \in H: yP_i x\}$. It is now a simple matter to define a _fair_ state x as one which is both equitable _and_ pareto-optimal. This leads naturally to a definition of

Fairness (Condition F): _Fairness_ requires that for all $x, y \in X$ such that x is fair and y is not fair, society should prefer x to y.

3. A RESULT ON FAIRNESS AND LIBERTY.

Theorem. There exists no ASDF which simultaneously satisfies conditions U, L^* and F.

Proof. Let $\{x, y, z, w\}$ be the designated agenda, with individual 1 dictating on (x,y) and individual 2 dictating on (z,w). One must distinguish two cases -- one in which the pairs (x,y) and (z,w) have exactly one alternative in common (say y=z), and a second in which all of the four alternatives x,y,z,w are distinct.

Case i. Condition U sanctions the following preference configuration: wP_1xP_1z and xP_2zP_2w. By condition L^*, xPz and

zPw, so that acyclicity of the social preference relation leads to xRw. Assume now that w is equitable and x is not. Given the preference conflict over the pair (x,w), it is clear that w is also pareto optimal. Hence w is fair and x is not, so that condition F dictates wPx which contradicts xRw derived earlier.

Case ii. Invoking condition U again, the following preferences can be postulated for individuals 1 and 2: xP_1y, yP_1z, xP_1w and zP_2y, zP_2w, wP_2x. By condition L*, xPy and zPw. Let y and w be equitable, and x and z not. In view of the preference conflicts over (y,z) and (x,w), y and w can be seen to be pareto optimal (in addition to being equitable). Hence y is fair and z is not, and likewise w is fair and x is not. By condition F, yPz and wPx. To summarize, we now have: xPy, yPz, zPw, wPx -- which is a cyclical social preference sequence. This completes the proof of the theorem. //

4. AN EXAMPLE.

While the proof of the theorem above is simple, what is somewhat more difficult is to demonstrate that there are motivationally plausible individual preferences that support the result derived in section 3. An example, which covers case i of the theorem above, is provided below. The illustration entails an extension of Sen's (1970) 'prude-lewd' example, and revolves around allowing an individual to dictate on a pair of states in which more than just

one personal feature vary.

Consider a situation in which there are two smokers -- designated 1 and 2 respectively -- of whom the first person is a 'prude' and the second a 'lewd'. For $i=1,2$, let r_i stand for 'i reads Lady Chatterly's Lover', and s_i for 'i smokes'. Let x,y,w be three social states, described by:

x: $\langle(r_1,s_1),(\sim r_2,s_2)\rangle$;

y: $\langle(r_1,\sim s_1),(\sim r_2,s_2)\rangle$; and

w: $\langle(r_1,\sim s_1),(r_2,\sim s_2)\rangle$.

Clearly, the states (x,y) are in the protected sphere of 1, and the states (y,w) in that of 2. Since peson 1 prefers smoking to not smoking, we must have xPy by condition L^*. Person 2 prefers y to w, from the principle that 'a book's only a book, but a good cigar is a smoke', so we must have yPw by liberty to 2. Acyclicity dictates xRw. In state x, the prude (who reads LCL) envies the lewd (who does not read LCL), and the lewd likewise envies the prude. Hence x is not equitable, and therefore not fair. By contrast, in state w, both persons are identically situated: neither envies the other, so that state w is equitable. Further, assume that xP_1w (this follows from a straightforward preference of the prude's for smoking to not smoking), and wP_2x (the lewd is willing to forego the pleasures of smoking if this means he can read LCL and that the prude too must be deprived of a smoke: one could imagine that the lewd has a certain

concern for the prude's physical well-being). Given the preference conflict over (x,w), w is pareto optimal. Since w is also equitable, w must be fair -- unlike x. Condition F therefore dictates wPx, which however contradicts xRw, derived earlier.

5. CONCLUDING OBSERVATIONS.

If one accepts the Arrovian approach of a 'constitution' to the problem of social choice, then one must reckon with the disturbing possibility that liberal values may conflict with the desideratum of allocative fairness. This may or may not be a surprising result; at the formal level, however, it should be noted that the result is not logically entailed by the 'impossibility of a paretian liberal' result.

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RIGHTS AND PARETIANISM IN A FRAMEWORK OF FUZZY PREFERENCES

S.Subramanian*

ABSTRACT: In a recent paper, Bhaskar Dutta (2) considers the implications for Arrow-type impossibility theorems of relaxing the requirement that individual and social preferences be exact. He reports a fair measure of success in avoiding the Arrow paradox in a framework of fuzzy preferences. A similar exercise has been undertaken in the present paper in the context of Sen's liberal paradox, but the results are far from encouraging: the 'impossibility of a Paretian liberal' result seems to survive a fuzzy framework no better than it does an exact framework.

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RIGHTS AND PARETIANISM IN A FRAMEWORK OF FUZZY PREFERENCES

INTRODUCTION.

In a recent paper, Bhaskar Dutta (2) has explored the implications for Arrow-type impossibility results of working within a collective choice framework in which both individual and social preferences are regarded as being fuzzy, rather than exact. He demonstrates that the Arrow impossibility can be escaped in a fuzzy framework and that, with a sufficiently weak notion of collective rationality, one can even have a fuzzy preference aggregation mechanism which satisfies the Arrovian cinditions and the condition of 'positive responsiveness'. (This is an encouraging departure from the pessimism of the corresponding Mas-Collel -- Sonnenschein (5) impossibility result in the exact framework). However, Dutta suggests that insofar as social choice must of necessity be exact, the benefits -- in terms of escaping Arrow-type paradoxes -- of working within a framework of fuzzy preferences could quite well be suspect.

In the present paper -- which relies heavily on (2) -- I explore the implications for Amartya Sen's (7) liberal paradox of relaxing the traditional requirement that individual and social preferences be exact. The results of this preliminary exercise seem to suggest thst there is very little to be gained from the transition from an exact to a fuzzy framework: a fuzzy Paretian liberal is not less impossible than his exact cousin.

1. SOME ELEMENTS OF A FUZZY PREFERENCE FRAMEWORK.

X is a set of all available social states, or alternatives, with $/X/ \varepsilon [3, \infty)$. A fuzzy subset A of X is a functional relation $A: X \to [0,1]$ such that for all $x \varepsilon X$: $0 \leq A(x) \leq 1$.

A fuzzy weak binary preference relation (FWBPR) on X is a functional relation $R: X \times X \to [0,1]$ such that for all $x, y \varepsilon X$: $0 \leq R(x,y) \leq 1$. (Note that an exact weak binary preference relation on X would be a functional relation $R: X \times X \to \{0,1\}$ such that for all $x, y \varepsilon X$: $R(x,y) = 0$ or 1). Given an FWBPR R on X, its asymmetric and symmetric parts P and I can be defined, respectively, as:

$$\text{for all } x, y \varepsilon X: P(x,y) = \begin{cases} R(x,y) & \text{if } R(x,y) > R(y,x) \\ 0 & \text{otherwise} \end{cases}$$

$$I(x,y) = \min (R(x,y), R(y,x)).$$

For a rationalization of the above definitions of P and I, the reader is referred to proposition 2.5 in (2).

An FWBPR R on X is reflexive if and only if for all $x \varepsilon X$: $R(x,x) = 1$, and complete if and only if for all $x, y \varepsilon X$: $R(x,y) + R(y,x) \geq 1$. A fuzzy weak binary preference ordering (FWBPO) is a reflexive, complete and transitive FWBPR. The notion of transitivity of an FWBPR is not a unique one: two transitive relations discussed in (2) are what are referred to as T_1- and T_2- transitive relations.

An FWBPR R on X is T_1-transitive if and only if for all

distinct $x,y,z \in X$: $R(x,z) \geq \min\{R(x,y),R(y,z)\}$.
An FWBPR R on X is T_2-transitive if and only if for all distinct $x,y,z \in X$: $R(x,z) \geq R(x,y) + R(y,z) - 1$.
Both notions of transitivity above are special cases of a more general notion (Ovchinnikov(6) : cited and discussed in (2)) called max-star transitivity which is defined in terms of a binary operation * on the interval $[0,1]$ as follows: for all distinct $x,y,z \in X$: $R(x,z) \geq R(x,y) * R(y,z)$.

A crucial distinction between T_1-transitivity and T_2-transitivity resides in the implications which these two types of transitivity of R have for PP-transitivity of R. An FWBPR R on X is said to satisfy PP-transitivity of R if and only if for all distinct $x,y,z \in X$: $P(x,z) \geq P(x,y)*P(y,z)$.

The following two propositions are now true:

Proposition 1. T_1-transitivity of R implies PP-transitivity of R.

Proof. See (2): Proposition 2.9.2. //

Proposition 2. T_2-transitivity of R does not imply PP-transitivity of R.

Proof. See (2): Proposition 2.9.3. //

As pointed out in (2), the distinction between transitive relations which respectively do and do not imply PP-transitivity is of crucial importance in drawing a dividing line between possibility and impossibility results for some Arrow-type theorems in a fuzzy framework. This distinction is one which

-- as we shall see later -- is of no great significance for the liberal paradox (except when we base exact choice on fuzzy preferences).

We end this section with some concepts and notation which latter is, for purposes of consistency, the same as in (2). $N = \{1,...,i,...,n\}$ is a set of all individuals constituting society, with $n \geq 2$. If T is the set of all reflexive and complete FWBPRs on X, and T^n is the n-fold Cartesian product of T, then a fuzzy aggregation rule (FAR) is a functional relation $f: T^n \rightarrow T$. H_1 (respectively, H_2) will stand for the set of all FWBPOs on X which satisfy T_1- (respectively, T_2-) transitivity. By a fuzzy social decision function (FSDF) will be meant an FAR whose domain is H_1^n and whose range is H_1 or H_2; thus, for example, an FSDF $f: H_1^n \rightarrow H_2$ is an FAR which for every n-tuple $\{R_i\}$ of reflexive, complete and T_1-transitive FWBPRs in its domain specifies a unique reflexive, complete and T_2-transitive FWBPR R in its range. Our concern in this paper will be with the existence of an f which satisfies the ethical principles of paretianism and private liberty.

2. LIBERTY AND PARETO IN A FUZZY PREFERENCE FRAMEWORK.

Two conditions which one can impose on an FAR f are the following.

(a) The Pareto Condition (PC), which is satisfied if and only if for all distinct $x,y \in X$: $P(x,y) \geq \min_i P_i(x,y)$;

and

(b) Minimal Liberalism-1 (ML-1)[1], which is satisfied if and only if there exist at least two distinct individuals j and k and two distinct doubletons $D_j = \{x,y\}$ and $D_k = \{w,z\}$ of alternatives such that

(i) $P_j(x,y) > 0$ (resp., $P_j(y,x) > 0$) $\rightarrow$ $P(x,y) > 0$ (resp., $P(y,x) > 0$); and

(ii) $P_k(w,z) > 0$ (resp., $P_k(z,w) > 0$) $\rightarrow$ $P(w,z) > 0$ (resp., $P(z,w) > 0$).

The following proposition is now true.

Proposition 3. There exists no FSDF $f: H_1^n \rightarrow H_1$ which simultaneously satisfies conditions PC and ML-$\bar{1}$.

Proof. Consider the case where the doubletons $D_j=\{x,y\}$ and $D_k=\{w,z\}$ have exactly one alternative in common, say $y = w$. To avoid tedious repetition, we do not here take up for proof the case where $D_j \cap D_k = \emptyset$; for a complete proof, see Proposition 4 (to follow).

Let the following be the configuration of preferences of individual j and of all individuals other than j over the triple $\{x,y,z\}$:

j's FWBPO over $\{x,y,z\}$	FWBPO over $\{x,y,z\}$ of all $i \in N \setminus \{j\}$
$R_j(x,y) = \lambda \in (1/2,1)$	$R_i(x,y) = 1-\lambda$
$R_j(y,x) = 1-\lambda$	$R_i(y,x) = \lambda$
$R_j(y,z) = 1-\lambda$	$R_i(y,z) = \lambda$
$R_j(z,y) = \lambda$	$R_i(z,y) = 1-\lambda$
$R_j(x,z) = 1-\lambda$	$R_i(x,z) = 1-\lambda$
$R_j(z,x) = 1$	$R_i(z,x) = 1$

(It can be verified that for all $i \in N$, the R_i are complete and T_1-transitive). By ML-1 to person j over the pair (x,y), $P(x,y) > 0$. By ML-1 to person k over the pair (y,z), $P(y,z) > 0$. Since T_1-transitivity of R implies also PP-transitivity of R, we must have $P(x,z) > 0$, i.e., $P(z,x) = 0$. But by PC over the pair (z,x), $P(z,x) > 0$, which entails a contradiction.//

It may be imagined that the impossibility result derived above arises because T_1-transitivity of R implies also PP-transitivity of R, and that the impossibility could be avoided by expanding the range of f to include FWBPOs that satisfy T_2-transitivity. A weakening along this line does help in transforming an impossibility into a possibility result for certain Arrow-type theorems (see, in particular, Propositions 3.12 and 3.14 in (2)). However, as Proposition 4 below will show, no such relief is available in the case of the liberty paradox.

Proposition 4. There exists no FSDF $f: H_1^n \rightarrow H_2$ which simultaneously satisfies conditions PC and ML-1.

Proof. Again two cases must be distinguished: (i) the doubletons $D_j = \{x,y\}$ and $D_k = \{w,z\}$ have exactly one alternative in common (say $y = w$); and (ii) $\{x,y\} \cap \{w,z\} = \emptyset$.

Case (i). Using exactly the same configuration of individual preferences employed in the proof of proposition 3, we have, by ML-1 to person j over the pair (x,y), $P(x,y) = R(x,y) \overset{=}{-}$ (say) $\sigma_1 > 0$. Similarly, by ML-1 to person k

over the pair (y,z), $P(y,z) = R(y,z) \doteqdot$ (say) $\sigma_2 > 0$. By PC over the pair (z,x), $P(z,x) = R(z,x) = 1$. By T_2-transitivity over the triple $\{y,z,x\}$, $R(y,x) \geqq R(y,z) + R(z,x) - 1$, i.e., $R(y,x) \geqq \sigma_2$. Since $P(x,y) > 0$, we must have $R(x,y) > R(y,x)$ which entails, as a necessary condition, that

$$\sigma_1 > \sigma_2 \ . \qquad (1)$$

By T_2-transitivity over the triple $\{z,x,y\}$, $R(z,y) \geqq R(z,x) + R(x,y) - 1$, i.e., $R(z,y) \geqq \sigma_1$. Since $P(y,z) > 0$, we must have $R(y,z) > R(z,y)$, which entails, as a necessary condition, that

$$\sigma_2 > \sigma_1 \ . \qquad (2)$$

From (1) and (2) we obtain a contradiction.

Case (ii). Let the following be the configuration of individual preferences over $\{x,y,w,z\}$.

j's FWBPO over $\{x,y,w,z\}$: $R_j(x,y) = \lambda \in (1/2,1)$, $R_j(y,x) = 1-\lambda$; $R_j(y,z) = 1$, $R_j(z,y) = \lambda$; $R_j(w,x) = 1$, $R_j(x,w) = \lambda$; $R_j(x,z) = \lambda$, $R_j(z,x) = 1-\lambda$; $R_j(y,w) = 1-\lambda$, $R_j(w,y) = \lambda$; $R_j(w,z) = \lambda$, $R_j(z,w) = 1-\lambda$.

FWBPO over $\{x,y,w,z\}$ of all $i \in N \setminus \{j\}$: $R_i(x,y) = 1-\lambda$, $R_i(y,x) = \lambda$; $R_i(y,z) = 1$, $R_i(z,y) = \lambda$; $R_i(w,x) = 1$, $R_i(x,w) = \lambda$; $R_i(x,z) = 1-\lambda$, $R_i(z,x) = \lambda$; $R_i(y,w) = \lambda$, $R_i(w,y) = 1-\lambda$; $R_i(w,z) = 1-\lambda$, $R_i(z,w) = \lambda$.

(It is, again, easy to verify that the individual R_i are both complete and T_1-transitive). By ML-1 to person j over the

pair (x,y), $P(x,y) = R(x,y) =$ (say) $\mu_1 > 0$, and by ML-1 to person k over the pair (z,w), $P(z,w) = R(z,w) =$ (say) $\mu_2 > 0$. By PC over the pairs (y,z) and (w,x) respectively, $P(y,z) = R(y,z) = 1$ and $P(w,x) = R(w,x) = 1$. By T_2-transitivity over the triple $\{y,w,x\}$, $R(y,x) \geqq R(y,w) + R(w,x) - 1$, i.e., $R(y,x) \geqq R(y,w)$. By T_2-transitivity over $\{y,z,w\}$, $R(y,w) \geqq R(y,z) + R(z,w) - 1$, i.e., $R(y,w) \geqq R(z,w)$. We thus have $R(y,x) \geqq R(y,w) \geqq R(z,w) = \mu_2$. Since $P(x,y) > 0$, $R(x,y) > R(y,x)$, which entails, as a necessary condition, that

$$\mu_1 > \mu_2 \quad (3)$$

Further, by T_2-transitivity over the triple $\{w,x,z\}$, $R(w,z) \geqq R(w,x) + R(x,z) - 1$, i.e., $R(w,z) \geqq R(x,z)$. Again by T_2-transitivity over $\{x,y,z\}$, $R(x,z) \geqq R(x,y) + R(y,z) - 1$, i.e., $R(x,z) \geqq R(x,y)$. We thus have $R(w,z) \geqq R(x,z) \geqq R(x,y) = \mu_1$. Since $P(z,w) > 0$, we must have $R(z,w) > R(w,z)$ so that, necessarily,

$$\mu_2 > \mu_1 \quad (4)$$

(3) and (4) are mutually incompatible. This completes the proof of the proposition. //

In order to secure a possibility result, it is clear that the liberty axiom needs to be weakened. A weakened version of condition ML-1 is embodied in

Minimal Liberalism-2 (ML-2), which is satisfied if and only if there exist at least two individuals j and k and two distinct doubletons $D_j = \{x,y\}$ and $D_k = \{w,z\}$ of alternatives such that

(i) $P_j(x,y) > 0$ (resp., $P_j(y,x) > 0$) $\rightarrow P(y,x) = 0$ (resp.,
$P(x,y) = 0$) ;

and

(ii) $P_k(w,z) > 0$ (resp., $P_k(z,w) > 0$) $\rightarrow P(z,w) = 0$ (resp.,
$P(w,z) = 0$).

In effect, while ML-1 permits each of at least two individuals to 'dictate' on the alternatives in his protected sphere, ML-2 permits each person only 'the right of veto' on the alternatives in his protected sphere. This weakening of the liberty principle corresponds to Karni's (4) weakening of Sen's (7) liberty axiom in the exact framework, whereby individuals are allowed only semidecisiveness rather than 'full' decisiveness over the alternatives in their respective protected spheres. Such an anaemic notion of individual rights now secures a possibility result.

Proposition 5. There exists an FSDF $f: H_1^n \rightarrow H_1$ which satisfies conditions PC and ML-2.

Proof. Construct the following FAR $\tilde{f}$:

$\forall\ x,y \in X,\ \forall \{R_i\}_{i\in N} \in H_1^n : \tilde{R}(x,x) = 1;$

$\forall x,y\ (x \neq y) \in X,\ \forall \{R_i\}_{i\in N} \in H_1^n :$

$$\tilde{R}(x,y) = \begin{cases} 1 \text{ if } /S(x,y)/ = n \\ 1/2 \text{ otherwise} \end{cases}$$

where for all distinct $x,y \in X$: $S(x,y) \equiv \{i \in N / R_i(x,y) > R_i(y,x)\}$, and $/S(x,y)/$ denotes the cardinality of the set $S(x,y)$.

Note first that for all distinct $x,y \in X$: min $(\tilde{R}(x,y), \tilde{R}(y,x))$

$= 1/2$, which ensures completeness of $\tilde{R}$. That $\tilde{f}$ satisfies PC is obvious: if $\min_i P_i(x,y) = 0$, then clearly $/S(x,y)/ \leq n$, so that $\tilde{R}(x,y) = 1/2$ and $\tilde{R}(y,x) \geq 1/2$, entailing $\tilde{P}(x,y) = 0$; if $\min_i P_i(x,y) > 0$, then $/S(x,y)/ = n$ and $/S(y,x)/ = 0$, so that, with $\tilde{R}(x,y) = 1 > \tilde{R}(y,x) = 1/2$ we must have $\tilde{P}(x,y) = \tilde{R}(x,y) = 1 \geq \min_i R_i(x,y)$.

To see that $\tilde{f}$ satisfies ML-2, note that if the pair (x,y) constitutes the protected sphere of some individual j, and $R_j(x,y) > R_j(y,x)$, then, since $/S(x,y)/ \leq n$, we must have $\tilde{R}(y,x) = 1/2$; further, $\tilde{R}(x,y) \geq 1/2$, so $\tilde{P}(y,x) = 0$ as required for the fullfilment of ML-2.

It only remains to demonstrate that $\tilde{R}$ is T_1-transitive, i.e., that given a triple $\{x,y,z\}$, $\tilde{R}(x,z) \geq \min(\tilde{R}(x,y),\tilde{R}(y,z))$. It is elementary to note that $\tilde{R}$ satisfies T_1-transitivity when $\min(\tilde{R}(x,y),\tilde{R}(y,z)) = 1/2$. In the case where $\tilde{R}(x,y) = \tilde{R}(y,z) = 1$, we must have $/S(x,y)/ = /S(y,z)/ = n$: since T_1-transitivity of the individual R_i implies PP-transitivity of the R_i, it must be the case that $/S(x,z)/ = n$, so that, by construction of $\tilde{f}$, $\tilde{R}(x,z) = 1$. Proposition 5 is thus proved. //

It is routine to note that the above result will hold when the range of f is expanded to include T_2-transitive FWBPOs, i.e., that there exists an FSDF $f{:}H_1^n \rightarrow H_2$ which satisfies conditions PC and ML-2.

3. CHOICE FUNCTIONS AND THE POSSIBILITY OF A PARETIAN LIBERAL.

As was noted in the Introduction to this paper, social choice is constrained to be exact even though it may be based on preferences that are inexact, or fuzzy. It therefore becomes important to examine if the existence result embodied in Proposition 5 of the previous section carries through when we adopt a choice functional approach to the aggregation problem.

Let Σ be the null-set-excluded power set of X, the set of all social states. Every $A \in \Sigma$ is an agenda. An exact choice function (ECF) is a functional relation $C: \Sigma \rightarrow \Sigma$ such that for every $A \in \Sigma$, $C(A)\ (\neq \{\emptyset\}) \subseteq A$, that is, an ECF is a functional relation which, for every non-empty agenda A of Σ specifies a non-empty subset C(A) of A; C(A) is the set of chosen alternatives, or the choice set.

Given an FWBPR R and $A \in \Sigma$, the set of R-greatest alternatives in A is described as a functional relation $G(A,R): A \rightarrow [0,1]$ such that for all $x \in A$: $G(A,R)(x) = \min_{y \in A} R(x,y)$. Thus $G(A,R)(x)$ is 'the extent to which x belongs to the R-greatest set of A'. x will be said to belong to the choice set of A if and only if the extent to which x belongs to the R-greatest set of A is not less than the extent to which every other alternative in A belongs to its R-greatest set, that is, $x \in C(A) \leftrightarrow G(A,R)(x) \geq G(A,R)(y)\ \forall y \in A$.

Under the above condition, an ECF C will be said to be '<u>H-rationalizable</u>' in terms of an FWBPR R (see Dutta(2)). An ECF C will be said to be <u>H-generated</u> by an FAR $f:T^n \rightarrow T$ if for all $\{R_i\}_{i\varepsilon N} \varepsilon T^n$, C is rationalizable in terms of the FWBPR $f(\{R_i\}_{i\varepsilon N})$.

The following proposition is now true.

<u>Proposition 6.</u> Given an FSDF $f:H_1^n \rightarrow H_2$ which satisfies conditions PC and ML-2, and an ECF C which is H-generated by f, there exists an agenda $A\varepsilon\Sigma$ such that if any Pareto optimal alternative in A is in the choice set of A, then <u>every</u> Pareto optimal alternative in A must be in the choice set of A.

<u>Proof.</u>[2] Let $A = \{x,y,z\}$ be the designated agenda, with the pair (x,y) constituting the protected sphere of person j and the pair (y,z) that of persom k. Consider the configuration of individual preferences invoked in the proof of Proposition 3. It will be noted that x is Pareto-dominated by z, but y and z are Pareto optimal. Our object will be to demonstrate that $y\varepsilon C(A) \leftrightarrow z\varepsilon C(A)$.

Suppose, first, that $y \notin C(A)$ and $z\varepsilon C(A)$. Since $z\varepsilon C(A)$, we must have $G(A,R)(z) \geq G(A,R)(y)$ by virtue of the fact that C is H-generated by R, i.e., $\min(R(z,x),R(z,y)) \geq \min(R(y,x),R(y,z))$, i.e.,

$$R(z,y) \geq \min(R(y,x),R(y,z)) \qquad (1)$$

since $R(z,x) = 1$ by PC over the pair (z,x), and $1 \geq R(z,y)$.

Now, by T_2-transitivity over the triple $\{y,z,x\}$, and noting that $R(z,x) = 1$,

$$R(y,x) \geq R(y,z). \qquad (2)$$

From (1) and (2),

$$R(z,y) \geq R(y,z). \qquad (3)$$

But by ML-2 to person k over the pair (y,z), $R(y,z) \geq R(z,y)$; putting this together with (3) yields $R(y,z) = R(z,y)$ so that

$$G(A,R)(y) = G(A,R)(z). \qquad (4)$$

Further, $z \in C(A)$ implies $G(A,R)(z) \geq G(A,R)(x)$ which, in view of (4), implies

$$G(A,R)(y) \geq G(A,R)(x). \qquad (5)$$

From (4) and (5), we obtain $y \in C(A)$ contrary to supposition. Thus, if $y \notin C(A)$, then $z \notin C(A)$ or, equivalently,

$$z \in C(A) \rightarrow y \in C(A). \qquad (6)$$

Suppose, next, that $y \in C(A)$ and $z \notin C(A)$. Since $y \in C(A)$, $G(A,R)(y) \geq G(A,R)(z)$, i.e., $\min(R(y,x),R(y,z)) \geq \min(R(z,x),R(z,y))$, i.e.,

$$G(A,R)(y) = R(y,z) \geq R(z,y) = G(A,R)(z) \qquad (7)$$

since $R(z,x) = 1$ by PC over the pair (z,x) and $R(y,x) \geq R(y,z)$ by (2). By T_2-transitivity over the triple $\{z,x,y\}$, and noting that $R(z,x) = 1$,

$$R(z,y) \geq R(x,y). \qquad (8)$$

By ML-2 to person j over the pair (x,y), $R(x,y) \geq R(y,x)$; so that given (8),

$$R(z,y) \geq R(y,x). \qquad (9)$$

But, as already noted, $R(y,x) \geqq R(y,z)$, so from (9) we have:

$$R(z,y) \geqq R(y,z). \tag{10}$$

From (7) and (10) we obtain:

$$G(A,R)(y) = G(A,R)(z). \tag{11}$$

Further, since $y \varepsilon C(A)$, $G(A,R)(y) \geqq G(A,R)(x)$ which, in view of (11), implies

$$G(A,R)(z) \geqq G(A,R)(x). \tag{12}$$

(11) and (12) together imply that $z \varepsilon C(A)$ contrary to supposition. We have thus proved that

$$y \varepsilon C(A) \rightarrow z \varepsilon C(A). \tag{13}$$

From (6) and (13), we obtain the desired result:

$y\ C(A) \leftrightarrow z\ C(A)$. //

The result above shows that any T_2-transitive fuzzy aggregation rule f that satisfies conditions PC and ML-2 must be a peculiar one: if f admits any Pareto optimal state in the choice set, it must admit <u>every</u> Pareto optimal state in the choice set. As a practical guide to collective choice, such a rule must surely be regarded as being less than useful.

If the implications for exact social choice are embarrassing when the FAR f effects a mapping from H_1^n to H_2, they are even more serious when f effects a mapping from H_1^n to H_1. Before coming to that, we take note of a consistency condition on choice functions due to Bordes(1), called property β^+.

Property β^+: β^+ requires of an ECF C that for all $x,y \in X$ and all $S,T \in \Sigma$: $S \subseteq T$,

$(x \in C(S)$ and $y \in S) \rightarrow (y \in C(T) \rightarrow x \in C(T))$.

β^+ is the strongest of a class of 'expansion consistency' properties of choice functions that have been proposed in the literature, and for a comprehensive discussion of the subject in the exact framework, the reader is referred to Sen (9). The following lemma -- due to Dutta, Panda and Pattanaik (3), and cited in (2) -- is of importance for our next result.

Lemma. An ECF C which is H-rationalizable in terms of an $R \in H_1$ satisfies property β^+.

Proof. See (3). //

The following proposition is now true.

Proposition 7. Given an FSDF f: $H_1^n \rightarrow H_1$ which satisfies conditions PC and ML-2, and an ECF $\bar{C}$ which is H-generated by f, there exists an agenda $A \in \Sigma$ such that a Pareto-dominated alternative in A can be excluded from its choice set only at the cost of rendering the choice set null.

Proof. Let $A = \{x,y,z\}$ be the designated agenda, with the pair of alternatives (x,y) constituting the protected sphere of person j, and the pair (y,z) that of person k. We again invoke the configuration of individual preferences employed in the proof of Proposition 3. It can be seen that x is the only Pareto-dominated alternative (dominated by z) in A, and our object will be to show that if $x \notin C(A)$, then $C(A) = \{ \emptyset \}$.

Suppose, to the contrary, that $x \notin C(A)$ and $C(A) \neq \{\emptyset\}$. In particular, let it be the case that $y \in C(A)$. By ML-2 to persom j over the pair (x,y), $R(x,y) \geq R(y,x)$. But $R(x,y) = G(\{x,y\},R)(x)$ and $R(y,x) = G(\{x,y\},R)(y)$, and since $G(\{x,y\},R)(x) \geq G(\{x,y\{,R)(y)$, we must have $x \in C(\{x,y\})$. But by the lemma stated earlier, C satisfies property β^+, so $x \in C(\{x,y\})$ and $y \in C(A)$ must imply that $x \in C(A)$, contrary to supposition. To avoid contradiction, then, we must infer that if $x \notin C(A)$, then $y \notin C(A)$. Suppose next that $z \in C(A)$. By ML-2 to person k over the pair (y,z), $R(y,z) \geq R(z,y)$. But $R(y,z) = G(\{y,z\},R)(y)$ and $R(z,y) = G(\{y,z\},R)(z)$, and since $G(\{y,z\},R)(y) \geq G(\{y,z\},R)(z)$, we must have $y \in C(\{y,z\})$. Again, since C satisfies β^+, $y \in C(\{y,z\})$ and $z \in C(A)$ must imply that $y \in C(A)$, which however violates the inference -- already drawn -- that $y \notin C(A)$. We must conclude then that if $x \notin C(A)$, then $z \notin C(A)$. Briefly, $x \notin C(A)$ implies $C(A) = \{\emptyset\}$, as desired. //

To interpret the above results with greater specificity, it is useful to furnish an example. The handiest example is Sen's (7) celebrated 'prude-lewd' example. Let person j be a lewd and person k a prude, and let there be one copy of Lady Chatterly's Lover available. x is a state in which only the lewd reads LCL, y a state in which no-one reads the book, and z a state in which only the prude reads LCL. As was noted in Section 2, the existence theorem

(Proposition 5) is essentially the fuzzy counterpart of a possibility result presented by Karni(4) in the exact framework. To obtain an intuitive and informal flavour of the implications of Propositions 6 and 7, it is apposite to quote Sen(8, p.227) on the choice functional entailments of Karni's result in the exact framework:

It is worth examining Karni's ... interesting reformulation of liberalism. Essentially, he proposes that the libertarian right be weakened to being semi-decisive over a pair rather than decisive, i.e., if the pair containing x and y is assigned to a person, then whenever he strictly prefers x to y, x is judged to be socially at least as good as y ... . This weakened libertarianism is consistent with the Pareto principle in the sense of not producing strict preference cycles... . Social strict preference may be transitive, but if so, all Pareto non-comparable states may have to be declared socially indifferent... which will make for a very peculiar system indeed.
An even more serious difficulty with Karni's method of resolution lies in the fact that i being semidecisive for x against y guarantees only that x is chosen from that pair (though not necessarily uniquely), but it does not prevent y from being chosen with x rejected from larger sets containing both x and y. In the Lady Chatterly's Lover illustration, under Karni's rule [the lewd] will be able to read the book if the choice is between his reading it and no one reading it (both alternatives will be judged equally acceptable[3]), but not if the possibility of [the prude] reading it is also included. Indeed, thanks to the Pareto principle, kept intact by Karni, [the lewd] will definitely not be able to read the book when the choice is over the three alternatives. Thus semidecisiveness over a pair is a very limited right indeed, and it will be a very tame libertarian who will settle for it.

CONCLUDING REMARKS.

There is clearly more than one way of translating concepts in an exact framework to corresponding concepts in a fuzzy framework. The particular approach we have adopted in this paper in the context of the dilemma of the Paretian liberal

seems to suggest that the dilemma is not easily escaped by an appeal to the limitations of a two-valued logic. Life for a Paretian liberal, it would appear, is not noticeably more relaxed in a framework of fuzzy preferences than in its exact counterpart.

NOTES.

1. A case could be made for an even stronger libertarian principle which would consist in adding a third restriction to the two already contained in condition ML-1:

$$\text{(iii) } P(x,y) \gtreqless P(w,z) \text{ (resp., } P(y,x) \gtreqless P(z,w)) \leftrightarrow P_j(x,y) \gtreqless P_k(w,z) \text{ (resp., } P_j(y,x) \gtreqless P_k(z,w)).$$

Restriction (iii) above essentially requires social fuzzy preferences over alternatives in 'protected spheres' to move 'monotonically' with individual preferences.

A second libertarian principle, independent of the one above, but also stronger than condition ML-1, is one in which restrictions (i) and (ii) in the definition of ML-1 are replaced by:

$$\text{(i') } P_j(x,y) > 0 \text{ (resp., } P_j(y,x) > 0) \rightarrow P(x,y) \geqq P_j(x,y) \text{ (resp., } P(y,x) \geqq P_j(y,x)); \text{ and}$$

$$\text{(ii') } P_k(w,z) > 0 \text{ (resp., } P_k(z,w) > 0) \rightarrow P(w,z) \geqq P_k(w,z) \text{ (resp., } P(z,w) \geqq P_k(z,w)).$$

Needless to say, impossibility results would obtain a <u>fortiori</u> with these strengthened versions of liberalism.

2. In the proofs of both Propositions 6 and 7 we omit, for purposes of brevity, to consider the case where the agenda A contains four <u>distinct</u> alternatives.

3. This is true in a framework of fuzzy preferences as well. In the proof of Proposition 6, we have from (2) and (3): $R(y,x) \geqq R(z,y)$. By T_2-transitivity over the triple

$\{z,x,y\}$, $R(z,y) \geq R(z,x)+R(x,y)-1$, i.e., $R(z,y) \geq R(x,y)$ since $R(z,x)=1$. Hence, $R(y,x) \geq R(x,y)$. But $R(x,y) \geq R(y,x)$ by ML-2 to person j over the pair (x,y). Consequently, $R(x,y)=R(y,x)$, viz., $G(\{x,y\},R)(x)=G(\{x,y\},R)(y)$, entailing: $C(\{x,y\})=\{x,y\}$ -- that is, the alternatives of the lewd reading LCL and no-one reading it 'will be judjed equally acceptable.'

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